

Algebraic Number Theory

(PARI-GP version 2.11.0)

Binary Quadratic Forms

create $ax^2 + bxy + cy^2$ (distance d) `Qfb($a, b, c, \{d\}$)`
reduce x ($s = \sqrt{D}$, $l = \lfloor s \rfloor$) `qfbred($x, \{flag\}, \{D\}, \{l\}, \{s\}$)`
return $[y, g]$, $g \in \text{SL}_2(\mathbf{Z})$, $y = g \cdot x$ reduced `qfbreds12(x)`
composition of forms $x*y$ or `qfbnucomp(x, y, l)`
 n -th power of form x^n or `qfbnupow(x, n)`
composition without reduction `qfbcomprow(x, y)`
 n -th power without reduction `qfbpowrow(x, n)`
prime form of disc. x above prime p `qfbprimeform(x, p)`
class number of disc. x `qfbclassno(x)`
Hurwitz class number of disc. x `qfbhclassno(x)`
solve $Q(x, y) = p$ in integers, p prime `qfbsolve(Q, p)`

Quadratic Fields

quadratic number $\omega = \sqrt{x}$ or $(1 + \sqrt{x})/2$ `quadgen(x)`
minimal polynomial of ω `quadpoly(x)`
discriminant of $\mathbf{Q}(\sqrt{x})$ `quaddisc(x)`
regulator of real quadratic field `quadregulator(x)`
fundamental unit in real $\mathbf{Q}(\sqrt{D})$ `quadunit($D, \{w\}$)`
class group of $\mathbf{Q}(\sqrt{D})$ `quadclassunit($D, \{flag\}, \{t\}$)`
Hilbert class field of $\mathbf{Q}(\sqrt{D})$ `quadhilbert($D, \{flag\}$)`
... using specific class invariant ($D < 0$) `polclass($D, \{inv\}$)`
ray class field modulo f of $\mathbf{Q}(\sqrt{D})$ `quadray($D, f, \{flag\}$)`

General Number Fields: Initializations

The number field $K = \mathbf{Q}[X]/(f)$ is given by irreducible $f \in \mathbf{Q}[X]$. We denote $\theta = \bar{X}$ the canonical root of f in K . A nf structure contains a maximal order and allows operations on elements and ideals. A bnf adds class group and units. A bnr is attached to ray class groups and class field theory. A rnf is attached to relative extensions L/K .

init number field structure nf `nfinit($f, \{flag\}$)`
known integer basis B `nfinit($[f, B]$)`
order maximal at $vp = [p_1, \dots, p_k]$ `nfinit($[f, vp]$)`
order maximal at all $p \leq P$ `nfinit($[f, P]$)`
certify maximal order `nfcertify(nf)`

nf members:

a monic $F \in \mathbf{Z}[X]$ defining K $nf.pol$
number of real/complex places $nf.r1/r2/sign$
discriminant of nf $nf.disc$
 T_2 matrix $nf.t2$
complex roots of F $nf.roots$
integral basis of \mathbf{Z}_K as powers of θ $nf.zk$
different/codifferent $nf.diff, nf.codiff$
index $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$ $nf.index$
recompute nf using current precision `nfnewprec(nf)`
init relative rnf $L = K[Y]/(g)$ `rnfinit(nf, g)`
init bnf structure `bnfinit($f, \{flag\}$)`

bnf members:

same as nf , plus
underlying nf $bnf.nf$
classgroup $bnf.clgp$
regulator $bnf.reg$
fundamental/torsion units $bnf.fu, bnf.tu$

compress a bnf for storage `bnfcompress(bnf)`
recover a bnf from compressed $bnfz$ `bnfinit($bnfz$)`
add S -class group and units, yield $bnfS$ `bnfsunit(bnf, S)`
init class field structure bnr `bnrinit($bnf, m, \{flag\}$)`
bnr members: same as bnf , plus
underlying bnf $bnr.bnf$
big ideal structure $bnr.bid$
modulus $bnr.mod$
structure of $(\mathbf{Z}_K/m)^*$ $bnr.zkst$

Fields, subfields, embeddings

Defining polynomials, embeddings
smallest poly defining $f = 0$ (slow) `polredabs($f, \{flag\}$)`
small poly defining $f = 0$ (fast) `polredbest($f, \{flag\}$)`
random Tschirnhausen transform of f `poltschirnhaus(f)`
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$? Isomorphic? `nfisincl(f, g), nfisom`
reverse polmod $a = A(t) \bmod T(t)$ `modreverse(a)`
compositum of $\mathbf{Q}[t]/(f), \mathbf{Q}[t]/(g)$ `polcompositum($f, g, \{flag\}$)`
compositum of $K[t]/(f), K[t]/(g)$ `nfcompositum($nf, f, g, \{flag\}$)`
splitting field of K (degree divides d) `nfsplitting($nf, \{d\}$)`
signs of real embeddings of x `nfeltsign($nf, x, \{pl\}$)`
complex embeddings of x `nfeltembed($nf, x, \{pl\}$)`
 $T \in K[t]$, # of real roots of $\sigma(T) \in R[t]$ `nfpolsturm($nf, T, \{pl\}$)`

Subfields, polynomial factorization

subfields (of degree d) of nf `nfsubfields($nf, \{d\}$)`
 d -th degree subfield of $\mathbf{Q}(\zeta_n)$ `polsubcyclo($n, d, \{v\}$)`
roots of unity in nf `nfrootsof1(nf)`
roots of g belonging to nf `nfroots(nf, g)`
factor g in nf `nfactor(nf, g)`
factor $g \bmod$ prime pr in nf `nfactormod(nf, g, pr)`

Linear and algebraic relations

poly of degree $\leq k$ with root $x \in \mathbf{C}$ `algdep(x, k)`
alg. dep. with pol. coeffs for series s `seralgdep(s, x, y)`
small linear rel. on coords of vector x `lindep(x)`

Basic Number Field Arithmetic (nf)

Number field elements are `t_INT`, `t_FRAC`, `t_POL`, `t_POLMOD`, or `t_COL` (on integral basis $nf.zk$).

Basic operations

$x + y$ `nfeltadd(nf, x, y)`
 $x \times y$ `nfeltmul(nf, x, y)`
 $x^n, n \in \mathbf{Z}$ `nfeltpow(nf, x, n)`
 x/y `nfeltdiv(nf, x, y)`
 $q = x \backslash y := \text{round}(x/y)$ `nfeltdiveuc(nf, x, y)`
 $r = x \% y := x - (x \backslash y)y$ `nfeltmod(nf, x, y)`
... $[q, r]$ as above `nfeltdivrem(nf, x, y)`
reduce x modulo ideal A `nfeltreduce(nf, x, A)`
absolute trace $\text{Tr}_{K/\mathbf{Q}}(x)$ `nfelttrace(nf, x)`
absolute norm $N_{K/\mathbf{Q}}(x)$ `nfeltnorm(nf, x)`

Multiplicative structure of K^* ; $K^*/(K^*)^n$

valuation $vp_{\mathfrak{p}}(x)$ `nfeltval(nf, x, \mathfrak{p})`
... write $x = \pi^{vp_{\mathfrak{p}}(x)}y$ `nfeltval($nf, x, \mathfrak{p}, \&y$)`
quadratic Hilbert symbol (at \mathfrak{p}) `nfhilbert($nf, a, b, \{\mathfrak{p}\}$)`
 b such that $xb^n = v$ is small `idealredmodpower(nf, x, n)`

Maximal order and discriminant

integral basis of field $\mathbf{Q}[x]/(f)$ `nfbasis(f)`
field discriminant of field $f = 0$ `nfdisc(f)`
express x on integer basis `nfalgtobasis(nf, x)`
express element x as a polmod `nfbasistoalg(nf, x)`

Dedekind Zeta Function ζ_K , Hecke L series

$R = [c, w, h]$ in initialization means we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w$, $|\Im(s)| < h$; $R = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $R = [1/2, 0, h]$ (critical line up to height h).
 ζ_K as Dirichlet series, $N(I) < b$ `dirzetak(nf, b)`
init $\zeta_K^{(k)}(s)$ for $k \leq n$ `L = lfunitinit($bnf, R, \{n = 0\}$)`
compute $\zeta_K(s)$ (n -th derivative) `lfun($L, s, \{n = 0\}$)`
compute $\Lambda_K(s)$ (n -th derivative) `lfunlambda($L, s, \{n = 0\}$)`

init $L_K^{(k)}(s, \chi)$ for $k \leq n$ `L = lfunitinit($[bnr, chi], R, \{n = 0\}$)`
compute $L_K(s, \chi)$ (n -th derivative) `lfun($L, s, \{n\}$)`
Artin root number of K `bnrrootnumber($bnr, chi, \{flag\}$)`
 $L(1, \chi)$, for all χ trivial on H `bnrL1($bnr, \{H\}, \{flag\}$)`

Class Groups & Units (bnf, bnr)

Class field theory data $a_1, \{a_2\}$ is usually bnr (ray class field), bnr, H (congruence subgroup) or bnr, χ (character on `bnr.clgp`). Any of these define a unique abelian extension of K .
remove GRH assumption from bnf `bnfcertify(bnf)`
expo. of ideal x on class gp `bnfisprincipal($bnf, x, \{flag\}$)`
expo. of ideal x on ray class gp `bnrisprincipal($bnr, x, \{flag\}$)`
expo. of x on fund. units `bnfisunit(bnf, x)`
as above for S -units `bnfissunit($bnfs, x$)`
signs of real embeddings of $bnf.fu$ `bnfsignunit(bnf)`
narrow class group `bnfnarrow(bnf)`

Class Field Theory

ray class number for modulus m `bnrclassno(bnf, m)`
discriminant of class field `bnrdisc($a_1, \{a_2\}$)`
ray class numbers, l list of moduli `bnrclassnolist(bnf, l)`
discriminants of class fields `bnrdisclist($bnf, l, \{arch\}, \{flag\}$)`
decode output from `bnrdisclist` `bnfdecodemodule(nf, fa)`
is modulus the conductor? `bnrconductor($a_1, \{a_2\}$)`
is class field (bnr, H) Galois over K^G `bnrisgalois(bnr, G, H)`
action of automorphism on $bnr.gen$ `bnrgaloismatrix(bnr, aut)`
apply `bnrgaloismatrix` M to H `bnrgaloisapply(bnr, M, H)`
characters on `bnr.clgp` s.t. $\chi(g_i) = e(v_i)$ `bnrchar($bnr, g, \{v\}$)`
conductor of character χ `bnrconductor(bnr, chi)`
conductor of extension `bnrconductor($a_1, \{a_2\}, \{flag\}$)`
conductor of extension $K[Y]/(g)$ `rnfconductor(bnf, g)`
Artin group of extension $K[Y]/(g)$ `rnfnormgroup(bnr, g)`
subgroups of bnr , index $\leq b$ `subgrouplist($bnr, b, \{flag\}$)`
rel. eq. for class field def'd by sub `rnfkummer($bnr, sub, \{d\}$)`
same, using Stark units (real field) `bnrstark($bnr, sub, \{flag\}$)`
is a an n -th power in K_v ? `nfislocalpower(nf, v, a, n)`
cyclic L/K satisf. local conditions `nfgrunwaldwang(nf, P, D, pl)`

Logarithmic class group

logarithmic ℓ -class group `bnflog(bnf, ℓ)`
 $[\bar{e}(F_v/Q_p), \bar{f}(F_v/Q_p)]$ `bnflogef(bnf, pr)`
 $\exp \deg_F(A)$ `bnflogdegree(bnf, A, ℓ)`
is ℓ -extension L/K locally cyclotomic `rnfislocalcyclo(rnf)`

Ideals: elements, primes, or matrix of generators in HNF

is id an ideal in nf ? nfsideal(nf, id)
is x principal in bnf ? bnfisprincipal(bnf, x)
give $[a, b]$, s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$ idealtwoelt($nf, x, \{a\}$)
put ideal a ($a\mathbf{Z}_K + b\mathbf{Z}_K$) in HNF form idealhnf($nf, a, \{b\}$)
norm of ideal x idealnrm(nf, x)
minimum of ideal x (direction v) idealmin(nf, x, v)
LLL-reduce the ideal x (direction v) idealred($nf, x, \{v\}$)

Ideal Operations

add ideals x and y idealadd(nf, x, y)
multiply ideals x and y idealmul($nf, x, y, \{flag\}$)
intersection of ideals x and y idealintersect($nf, x, y, \{flag\}$)
 n -th power of ideal x idealpow($nf, x, n, \{flag\}$)
inverse of ideal x idealinv(nf, x)
divide ideal x by y idealdiv($nf, x, y, \{flag\}$)
Find $(a, b) \in x \times y, a + b = 1$ idealaddtoone($nf, x, \{y\}$)
coprime integral A, B such that $x = A/B$ idealnumden(nf, x)

Primes and Multiplicative Structure

factor ideal x in \mathbf{Z}_K idealfactor(nf, x)
expand ideal factorization in K idealfactorback($nf, f, \{e\}$)
is ideal A an n -th power ? idealispower(nf, A, n)
expand elt factorization in K nffactorback($nf, f, \{e\}$)
decomposition of prime p in \mathbf{Z}_K idealprimedec(nf, p)
valuation of x at prime ideal pr idealval(nf, x, pr)
weak approximation theorem in nf idealchinese(nf, x, y)
 $a \in K$, s.t. $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(x)$ if $v_{\mathfrak{p}}(x) \neq 0$ idealappr(nf, x)
 $a \in K$ such that $(a \cdot x, y) = 1$ idealcoprime(nf, x, y)
give bid =structure of $(\mathbf{Z}_K/id)^*$ idealstar($nf, id, \{flag\}$)
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$ idealprincipalunits(nf, pr, k)
discrete log of x in $(\mathbf{Z}_K/bid)^*$ ideallog(nf, x, bid)
idealstar of all ideals of norm $\leq b$ ideallist($nf, b, \{flag\}$)
add Archimedean places ideallistarch($nf, b, \{ar\}, \{flag\}$)
init **modpr** structure nfmodprinit(nf, pr)
project t to \mathbf{Z}_K/pr nfmodpr($nf, t, modpr$)
lift from \mathbf{Z}_K/pr nfmodprlift($nf, t, modpr$)

Galois theory over \mathbf{Q}

conjugates of a root θ of nf nfgaloisconj($nf, \{flag\}$)
apply Galois automorphism s to x nfgaloisapply(nf, s, x)
Galois group of field $\mathbf{Q}[x]/(f)$ polgalois(f)
initializes a Galois group structure G galoisinit($pol, \{den\}$)
character table of G galoischartable(G)
conjugacy classes of G galoisconjclasses(G)
 $\det(1 - \rho(g)T)$, χ character of ρ galoischarpoly($G, \chi, \{o\}$)
 $\det(\rho(g))$, χ character of ρ galoischarDET($G, \chi, \{o\}$)
action of p in nfgaloisconj form galoispermtopol($G, \{p\}$)
identify as abstract group galoisidentify(G)
export a group for GAP/MAGMA galoisexport($G, \{flag\}$)
subgroups of the Galois group G galoissubgroups(G)
is subgroup H normal? galoisisnormal(G, H)
subfields from subgroups galoissubfields($G, \{flag\}, \{v\}$)
fixed field galoisfixedfield($G, perm, \{flag\}, \{v\}$)
Frobenius at maximal ideal P idealfrobenius(nf, G, P)
ramification groups at P idealramgroups(nf, G, P)
is G abelian? galoisisabelian($G, \{flag\}$)
abelian number fields/ \mathbf{Q} galoissubcyclo($N, H, \{flag\}, \{v\}$)

Algebraic Number Theory

(PARI-GP version 2.11.0)

The galpol package

query the package: polynomial galoisgetpol(a,b,{s})
... : permutation group galoisgetgroup(a,b)
... : group description galoisgetname(a,b)

Relative Number Fields (rnf)

Extension L/K is defined by $T \in K[x]$. rnfequation($nf, T, \{flag\}$)
absolute equation of L rnfisabelian(nf, T)
is L/K abelian? rnfalgtobasis(rnf, x)
relative nfalgtobasis rnfbasistoalg(rnf, x)
relative nfbasistoalg rnfidealhnf(rnf, x)
relative idealhnf rnfidealmul(rnf, x, y)
relative idealmul rnfidealtwoelt(rnf, x)
relative idealtwoelt

Lifts and Push-downs

absolute \rightarrow relative representation for x rnfeltabstorel(rnf, x)
relative \rightarrow absolute representation for x rnfeltreltoabs(rnf, x)
lift x to the relative field rnfeltup(rnf, x)
push x down to the base field rnfeltdown(rnf, x)
idem for x ideal: (rnfideal)reltoabs, abstorel, up, down

Norms and Trace

relative norm of element $x \in L$ rnfeltnorm(rnf, x)
relative trace of element $x \in L$ rnfelttrace(rnf, x)
absolute norm of ideal x rnfidealnrmabs(rnf, x)
relative norm of ideal x rnfidealnrmrel(rnf, x)
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$ bnfisintnorm(bnf, x)
is $x \in \mathbf{Q}$ a norm from K ? bnfisnorm($bnf, x, \{flag\}$)
initialize T for norm eq. solver rnfisnorminit($K, pol, \{flag\}$)
is $a \in K$ a norm from L ? rnfisnorm($T, a, \{flag\}$)
initialize t for Thue equation solver thueinit(f)
solve Thue equation $f(x, y) = a$ thue($t, a, \{sol\}$)
characteristic poly. of $a \bmod T$ rnfcharpoly($nf, T, a, \{v\}$)

Factorization

factor ideal x in L rnfidealfactor(rnf, x)
 $[S, T]: T_{i,j} \mid S_i; S$ primes of K above p rnfidealprimedec(rnf, p)

Maximal order \mathbf{Z}_L as a \mathbf{Z}_K -module

relative polredbest rnfpolredbest(nf, T)
relative polredabs rnfpolredabs(nf, T)
relative Dedekind criterion, prime pr rnfdedekind(nf, T, pr)
discriminant of relative extension rnfdisc(nf, T)
pseudo-basis of \mathbf{Z}_L rnfpsseudobasis(nf, T)

General \mathbf{Z}_K -modules: $M = [\text{matrix, vec. of ideals}] \subset L$
relative HNF / SNF nfhnf(nf, M), nfsnf
multiple of det M nfdetint(nf, M)
HNF of M where $d = nfdetint(M)$ nfhnfmod(x, d)
reduced basis for M rnflllgram(nf, T, M)
determinant of pseudo-matrix M rnfDET(nf, M)
Steinitz class of M rnfsteinitz(nf, M)
 \mathbf{Z}_K -basis of M if \mathbf{Z}_K -free, or 0 rnfhnfbasis(bnf, M)
 n -basis of M , or $(n + 1)$ -generating set rnfBasis(bnf, M)
is M a free \mathbf{Z}_K -module? rnfisfree(bnf, M)

Associative Algebras

A is a general associative algebra given by a multiplication table mt (over \mathbf{Q} or \mathbf{F}_p); represented by al from **algtabinit**.
create al from mt (over \mathbf{F}_p) algtabinit($mt, \{p = 0\}$)
group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$) alggroup($G, \{p = 0\}$)
center of group algebra alggroupcenter($G, \{p = 0\}$)

Properties

is (mt, p) OK for **algtabinit**? algisassociative($mt, \{p = 0\}$)
multiplication table mt algmtable(al)
dimension of A over prime subfield algdim(al)
characteristic of A algchar(al)
is A commutative? algiscommutative(al)
is A simple? algissimple(al)
is A semi-simple? algissemisimple(al)
center of A algcenter(al)
Jacobson radical of A algradical(al)
radical J and simple factors of A/J algsimpledec(al)

Operations on algebras

create $A/I, I$ two-sided ideal algquotient(al, I)
create $A_1 \otimes A_2$ algtensor($al1, al2$)
create subalgebra from basis B algsubalg(al, B)
quotients by ortho. central idempotents e algcentralproj(al, e)
isomorphic alg. with integral mult. table algmakeintegral(mt)
prime subalgebra of semi-simple A over \mathbf{F}_p algprimesubalg(al)
find isomorphism $A \cong M_d(\mathbf{F}_q)$ algsplit(al)

Operations on lattices in algebras

lattice generated by cols. of M alglathnf(al, M)
... by the products $xy, x \in lat1, y \in lat2$ alglatmul($al, lat1, lat2$)
sum $lat1 + lat2$ of the lattices alglatadd($al, lat1, lat2$)
intersection $lat1 \cap lat2$ alglatinter($al, lat1, lat2$)
test $lat1 \subset lat2$ alglatsubset($al, lat1, lat2$)
generalized index $(lat2 : lat1)$ alglatindex($al, lat1, lat2$)
 $\{x \in al \mid x \cdot lat1 \subset lat2\}$ alglatlefttransporter($al, lat1, lat2$)
 $\{x \in al \mid lat1 \cdot x \subset lat2\}$ alglatrighttransporter($al, lat1, lat2$)
test $x \in lat$ (set $c = \text{coord. of } x$) alglatcontains($al, lat, x, \{&c\}$)
element of lat with coordinates c alglatelement(al, lat, c)

Operations on elements

$a + b, a - b, -a$ algadd(al, a, b), algsub, algneg
 $a \times b, a^2$ algmul(al, a, b), algsqr
 a^n, a^{-1} algpow(al, a, n), alginv
is x invertible ? (then set $z = x^{-1}$) algisinv($al, x, \{&z\}$)
find z such that $x \times z = y$ algdivl(al, x, y)
find z such that $z \times x = y$ algdivr(al, x, y)
does z s.t. $x \times z = y$ exist? (set it) algisdivl($al, x, y, \{&z\}$)
matrix of $v \mapsto x \cdot v$ algtomatrix(al, x)
absolute norm algnorm(al, x)
absolute trace algtrace(al, x)
absolute char. polynomial algcharpoly(al, x)
given $a \in A$ and polynomial T , return $T(a)$ algpoleval(al, T, a)
random element in a box algrandom(al, b)

Central Simple Algebras

A is a central simple algebra over a number field K ; represented by al from **alginit**; K is given by a nf structure.
create CSA from data **alginit**($B, C, \{v\}, \{maxord = 1\}$)
multiplication table over K $B = K, C = mt$
cyclic algebra ($L/K, \sigma, b$) $B = rnf, C = [sigma, b]$
quaternion algebra $(a, b)_K$ $B = K, C = [a, b]$
matrix algebra $M_d(K)$ $B = K, C = d$
local Hasse invariants over K $B = K, C = [d, [PR, HF], HI]$

Properties

type of al (mt, CSA) **algtype**(al)
dimension of A over \mathbf{Q} **algdim**($al, 1$)
dimension of al over its center K **algdim**(al)
degree of A ($= \sqrt{\dim_K A}$) **algdegree**(al)
 al a cyclic algebra ($L/K, \sigma, b$); return σ **algaut**(al)
...return b **algb**(al)
...return L/K , as an rnf **algsplittingfield**(al)
split A over an extension of K **algsplittingdata**(al)
splitting field of A as an rnf over center **algsplittingfield**(al)
multiplication table over center **algrelmultable**(al)
places of K at which A ramifies **algramifiedplaces**(al)
Hasse invariants at finite places of K **alghassef**(al)
Hasse invariants at infinite places of K **alghassei**(al)
Hasse invariant at place v **alghasse**(al, v)
index of A over K (at place v) **algindex**($al, \{v\}$)
is al a division algebra? (at place v) **algisdivision**($al, \{v\}$)
is A ramified? (at place v) **algisramified**($al, \{v\}$)
is A split? (at place v) **algissplit**($al, \{v\}$)

Operations on elements

reduced norm **algnorm**(al, x)
reduced trace **algtrace**(al, x)
reduced char. polynomial **algcharpoly**(al, x)
express x on integral basis **algalgtobasis**(al, x)
convert x to algebraic form **algbasistoalg**(al, x)
map $x \in A$ to $M_d(L)$, L split. field **algtomatrix**(al, x)

Orders

Z-basis of order \mathcal{O}_0 **algbasis**(al)
discriminant of order \mathcal{O}_0 **algdisc**(al)
Z-basis of natural order in terms \mathcal{O}_0 's basis **alginvbasis**(al)