

# Algebraic Number Theory

(PARI-GP version 2.11.0)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ )      `Qfb( $a, b, c, \{d\}$ )`  
reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ )      `qfbred( $x, \{flag\}, \{D\}, \{l\}, \{s\}$ )`  
return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced      `qfbreds12( $x$ )`  
composition of forms       $x*y$  or `qfbnucomp( $x, y, l$ )`  
 $n$ -th power of form       $x^n$  or `qfbnupow( $x, n$ )`  
composition without reduction      `qfbcomprow( $x, y$ )`  
 $n$ -th power without reduction      `qfbpowrow( $x, n$ )`  
prime form of disc.  $x$  above prime  $p$       `qfbprimeform( $x, p$ )`  
class number of disc.  $x$       `qfbclassno( $x$ )`  
Hurwitz class number of disc.  $x$       `qfbhclassno( $x$ )`  
solve  $Q(x, y) = p$  in integers,  $p$  prime      `qfbsolve( $Q, p$ )`

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$       `quadgen( $x$ )`  
minimal polynomial of  $\omega$       `quadpoly( $x$ )`  
discriminant of  $\mathbf{Q}(\sqrt{x})$       `quaddisc( $x$ )`  
regulator of real quadratic field      `quadregulator( $x$ )`  
fundamental unit in real  $\mathbf{Q}(\sqrt{D})$       `quadunit( $D, \{w\}$ )`  
class group of  $\mathbf{Q}(\sqrt{D})$       `quadclassunit( $D, \{flag\}, \{t\}$ )`  
Hilbert class field of  $\mathbf{Q}(\sqrt{D})$       `quadhilbert( $D, \{flag\}$ )`  
... using specific class invariant ( $D < 0$ )      `polclass( $D, \{inv\}$ )`  
ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$       `quadray( $D, f, \{flag\}$ )`

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ . We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A  $nf$  structure contains a maximal order and allows operations on elements and ideals. A  $bnf$  adds class group and units. A  $bnr$  is attached to ray class groups and class field theory. A  $rmf$  is attached to relative extensions  $L/K$ .

init number field structure  $nf$       `nfinit( $f, \{flag\}$ )`  
known integer basis  $B$       `nfinit( $[f, B]$ )`  
order maximal at  $vp = [p_1, \dots, p_k]$       `nfinit( $[f, vp]$ )`  
order maximal at all  $p \leq P$       `nfinit( $[f, P]$ )`  
certify maximal order      `nfcertify( $nf$ )`

### nf members:

a monic  $F \in \mathbf{Z}[X]$  defining  $K$        $nf.pol$   
number of real/complex places       $nf.r1/r2/sign$   
discriminant of  $nf$        $nf.disc$   
 $T_2$  matrix       $nf.t2$   
complex roots of  $F$        $nf.roots$   
integral basis of  $\mathbf{Z}_K$  as powers of  $\theta$        $nf.zk$   
different/codifferent       $nf.diff, nf.codiff$   
index  $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$        $nf.index$   
recompute  $nf$  using current precision      `nfnewprec( $nf$ )`  
init relative  $rmf$   $L = K[Y]/(g)$       `rnfininit( $nf, g$ )`  
init  $bnf$  structure      `bnfinit( $f, \{flag\}$ )`

### bnf members:

same as  $nf$ , plus  
underlying  $nf$        $bnf.nf$   
classgroup       $bnf.clgp$   
regulator       $bnf.reg$   
fundamental/torsion units       $bnf.fu, bnf.tu$

compress a  $bnf$  for storage      `bnfcompress( $bnf$ )`  
recover a  $bnf$  from compressed  $bnfz$       `bnfinit( $bnfz$ )`  
add  $S$ -class group and units, yield  $bnfS$       `bnfsunit( $bnf, S$ )`  
init class field structure  $bnr$       `bnrinit( $bnf, m, \{flag\}$ )`  
**bnr members:** same as  $bnf$ , plus  
underlying  $bnf$        $bnr.bnf$   
big ideal structure       $bnr.bid$   
modulus       $bnr.mod$   
structure of  $(\mathbf{Z}_K/m)^*$        $bnr.zkst$

## Fields, subfields, embeddings

**Defining polynomials, embeddings**  
smallest poly defining  $f = 0$  (slow)      `polredabs( $f, \{flag\}$ )`  
small poly defining  $f = 0$  (fast)      `polredbest( $f, \{flag\}$ )`  
random Tschirnhausen transform of  $f$       `poltschirnhaus( $f$ )`  
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$  ? Isomorphic?      `nfisincl( $f, g$ ), nfisismomodreverse( $a$ )`  
reverse polmod  $a = A(t) \bmod T(t)$   
compositum of  $\mathbf{Q}[t]/(f)$ ,  $\mathbf{Q}[t]/(g)$       `polcompositum( $f, g, \{flag\}$ )`  
compositum of  $K[t]/(f)$ ,  $K[t]/(g)$       `nfcompositum( $nf, f, g, \{flag\}$ )`  
splitting field of  $K$  (degree divides  $d$ )      `nfsplitting( $nf, \{d\}$ )`  
signs of real embeddings of  $x$       `nfeltsign( $nf, x, \{pl\}$ )`  
complex embeddings of  $x$       `nfeltembed( $nf, x, \{pl\}$ )`  
 $T \in K[t]$ , # of real roots of  $\sigma(T) \in R[t]$       `nfpolsturm( $nf, T, \{pl\}$ )`

### Subfields, polynomial factorization

subfields (of degree  $d$ ) of  $nf$       `nfsubfields( $nf, \{d\}$ )`  
 $d$ -th degree subfield of  $\mathbf{Q}(\zeta_n)$       `polsubcyclo( $n, d, \{v\}$ )`  
roots of unity in  $nf$       `nfrootsof1( $nf$ )`  
roots of  $g$  belonging to  $nf$       `nfroots( $nf, g$ )`  
factor  $g$  in  $nf$       `nfactor( $nf, g$ )`  
factor  $g \bmod$  prime  $pr$  in  $nf$       `nfactormod( $nf, g, pr$ )`

### Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       `algdep( $x, k$ )`  
alg. dep. with pol. coeffs for series  $s$       `seralgdep( $s, x, y$ )`  
small linear rel. on coords of vector  $x$       `lindep( $x$ )`

## Basic Number Field Arithmetic (nf)

Number field elements are `t_INT`, `t_FRAC`, `t_POL`, `t_POLMOD`, or `t_COL` (on integral basis  $nf.zk$ ).

### Basic operations

$x + y$       `nfeltadd( $nf, x, y$ )`  
 $x \times y$       `nfeltmul( $nf, x, y$ )`  
 $x^n$ ,  $n \in \mathbf{Z}$       `nfeltpow( $nf, x, n$ )`  
 $x/y$       `nfeltdiv( $nf, x, y$ )`  
 $q = x \backslash y := \text{round}(x/y)$       `nfeltdiveuc( $nf, x, y$ )`  
 $r = x \% y := x - (x \backslash y)y$       `nfeltmod( $nf, x, y$ )`  
...  $[q, r]$  as above      `nfeltdivrem( $nf, x, y$ )`  
reduce  $x$  modulo ideal  $A$       `nfeltreduce( $nf, x, A$ )`  
absolute trace  $\text{Tr}_{K/\mathbf{Q}}(x)$       `nfelttrace( $nf, x$ )`  
absolute norm  $N_{K/\mathbf{Q}}(x)$       `nfeltnorm( $nf, x$ )`

### Multiplicative structure of $K^*$ ; $K^*/(K^*)^n$

valuation  $v_p(x)$       `nfeltval( $nf, x, p$ )`  
... write  $x = \pi^{v_p(x)}y$       `nfeltval( $nf, x, p, \&y$ )`  
quadratic Hilbert symbol (at  $p$ )      `nfhilbert( $nf, a, b, \{p\}$ )`  
 $b$  such that  $xb^n = v$  is small      `idealredmodpower( $nf, x, n$ )`

### Maximal order and discriminant

integral basis of field  $\mathbf{Q}[x]/(f)$       `nfbasis( $f$ )`  
field discriminant of field  $f = 0$       `nfdisc( $f$ )`  
express  $x$  on integer basis      `nfalgtobasis( $nf, x$ )`  
express element  $x$  as a polmod      `nfbasistoalg( $nf, x$ )`

### Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).  
 $\zeta_K$  as Dirichlet series,  $N(I) < b$       `dirzetak( $nf, b$ )`  
init  $\zeta_K^{(k)}(s)$  for  $k \leq n$       `L = lfunitinit( $bnf, R, \{n = 0\}$ )`  
compute  $\zeta_K(s)$  ( $n$ -th derivative)      `lfun( $L, s, \{n = 0\}$ )`  
compute  $\Lambda_K(s)$  ( $n$ -th derivative)      `lfunlambda( $L, s, \{n = 0\}$ )`

init  $L_K^{(k)}(s, \chi)$  for  $k \leq n$       `L = lfunitinit( $[bnr, chi], R, \{n = 0\}$ )`  
compute  $L_K(s, \chi)$  ( $n$ -th derivative)      `lfun( $L, s, \{n\}$ )`  
Artin root number of  $K$       `bnrrootnumber( $bnr, chi, \{flag\}$ )`  
 $L(1, \chi)$ , for all  $\chi$  trivial on  $H$       `bnrL1( $bnr, \{H\}, \{flag\}$ )`

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually  $bnr$  (ray class field),  $bnr, H$  (congruence subgroup) or  $bnr, \chi$  (character on `bnr.clgp`). Any of these define a unique abelian extension of  $K$ .  
remove GRH assumption from  $bnf$       `bnfcertify( $bnf$ )`  
expo. of ideal  $x$  on class gp      `bnfisprincipal( $bnf, x, \{flag\}$ )`  
expo. of ideal  $x$  on ray class gp      `bnrisprincipal( $bnr, x, \{flag\}$ )`  
expo. of  $x$  on fund. units      `bnfisunit( $bnf, x$ )`  
as above for  $S$ -units      `bnfissunit( $bnfs, x$ )`  
signs of real embeddings of  $bnf.fu$       `bnfsignunit( $bnf$ )`  
narrow class group      `bnfnarrow( $bnf$ )`

### Class Field Theory

ray class number for modulus  $m$       `bnrclassno( $bnf, m$ )`  
discriminant of class field      `bnrdisc( $a_1, \{a_2\}$ )`  
ray class numbers,  $l$  list of moduli      `bnrclassnolist( $bnf, l$ )`  
discriminants of class fields      `bnrdisclist( $bnf, l, \{arch\}, \{flag\}$ )`  
decode output from `bnrdisclist`      `bnfdecodemodule( $nf, fa$ )`  
is modulus the conductor?      `bnrisconductor( $a_1, \{a_2\}$ )`  
is class field  $(bnr, H)$  Galois over  $K^G$       `bnrisgalois( $bnr, G, H$ )`  
action of automorphism on  $bnr.gen$       `bnrgaloismatrix( $bnr, aut$ )`  
apply `bnrgaloismatrix`  $M$  to  $H$       `bnrgaloisapply( $bnr, M, H$ )`  
characters on `bnr.clgp` s.t.  $\chi(g_i) = e(v_i)$       `bnrchar( $bnr, g, \{v\}$ )`  
conductor of character  $\chi$       `bnrconductor( $bnr, chi$ )`  
conductor of extension      `bnrconductor( $a_1, \{a_2\}, \{flag\}$ )`  
conductor of extension  $K[Y]/(g)$       `rnfconductor( $bnf, g$ )`  
Artin group of extension  $K[Y]/(g)$       `rnfnormgroup( $bnr, g$ )`  
subgroups of  $bnr$ , index  $\leq b$       `subgrouplist( $bnr, b, \{flag\}$ )`  
rel. eq. for class field def'd by  $sub$       `rnfkummer( $bnr, sub, \{d\}$ )`  
same, using Stark units (real field)      `bnrstark( $bnr, sub, \{flag\}$ )`  
is  $a$  an  $n$ -th power in  $K_v$  ?      `nfislocalpower( $nf, v, a, n$ )`  
cyclic  $L/K$  satisf. local conditions      `nfgrunwaldwang( $nf, P, D, pl$ )`

### Logarithmic class group

logarithmic  $\ell$ -class group      `bnflog( $bnf, \ell$ )`  
 $[\bar{e}(F_v/Q_p), \bar{f}(F_v/Q_p)]$       `bnflogef( $bnf, pr$ )`  
 $\exp \deg_F(A)$       `bnflogdegree( $bnf, A, \ell$ )`  
is  $\ell$ -extension  $L/K$  locally cyclotomic      `rnfislocalcyclo( $rmf$ )`

**Ideals:** elements, primes, or matrix of generators in HNF

is $id$ an ideal in $nf$ ?	<code>nfisideal(nf, id)</code>
is $x$ principal in $bnf$ ?	<code>bnfisprincipal(bnf, x)</code>
give $[a, b]$ , s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$	<code>idealtwoelt(nf, x, {a})</code>
put ideal $a$ ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form	<code>idealhnf(nf, a, {b})</code>
norm of ideal $x$	<code>idealnrm(nf, x)</code>
minimum of ideal $x$ (direction $v$ )	<code>idealmin(nf, x, v)</code>
LLL-reduce the ideal $x$ (direction $v$ )	<code>idealred(nf, x, {v})</code>

**Ideal Operations**

add ideals $x$ and $y$	<code>idealadd(nf, x, y)</code>
multiply ideals $x$ and $y$	<code>idealmul(nf, x, y, {flag})</code>
intersection of ideals $x$ and $y$	<code>idealintersect(nf, x, y, {flag})</code>
$n$ -th power of ideal $x$	<code>idealpow(nf, x, n, {flag})</code>
inverse of ideal $x$	<code>idealinv(nf, x)</code>
divide ideal $x$ by $y$	<code>idealdiv(nf, x, y, {flag})</code>
Find $(a, b) \in x \times y, a + b = 1$	<code>idealaddtoone(nf, x, {y})</code>
coprime integral $A, B$ such that $x = A/B$	<code>idealnumden(nf, x)</code>

**Primes and Multiplicative Structure**

factor ideal $x$ in $\mathbf{Z}_K$	<code>idealfactor(nf, x)</code>
expand ideal factorization in $K$	<code>idealfactorback(nf, f, {e})</code>
is ideal $A$ an $n$ -th power ?	<code>idealispower(nf, A, n)</code>
expand elt factorization in $K$	<code>nffactorback(nf, f, {e})</code>
decomposition of prime $p$ in $\mathbf{Z}_K$	<code>idealprimedec(nf, p)</code>
valuation of $x$ at prime ideal $pr$	<code>idealval(nf, x, pr)</code>
weak approximation theorem in $nf$	<code>idealchinese(nf, x, y)</code>
$a \in K$ , s.t. $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(x)$ if $v_{\mathfrak{p}}(x) \neq 0$	<code>idealappr(nf, x)</code>
$a \in K$ such that $(a \cdot x, y) = 1$	<code>idealcoprime(nf, x, y)</code>
give $bid$ =structure of $(\mathbf{Z}_K/id)^*$	<code>idealstar(nf, id, {flag})</code>
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$	<code>idealprincipalunits(nf, pr, k)</code>
discrete log of $x$ in $(\mathbf{Z}_K/bid)^*$	<code>ideallog(nf, x, bid)</code>
<b>idealstar</b> of all ideals of norm $\leq b$	<code>ideallist(nf, b, {flag})</code>
add Archimedean places	<code>ideallistarch(nf, b, {ar}, {flag})</code>
init <b>modpr</b> structure	<code>nfmodprinit(nf, pr)</code>
project $t$ to $\mathbf{Z}_K/pr$	<code>nfmodpr(nf, t, modpr)</code>
lift from $\mathbf{Z}_K/pr$	<code>nfmodprlift(nf, t, modpr)</code>

**Galois theory over  $\mathbf{Q}$**

conjugates of a root $\theta$ of $nf$	<code>nfgaloisconj(nf, {flag})</code>
apply Galois automorphism $s$ to $x$	<code>nfgaloisapply(nf, s, x)</code>
Galois group of field $\mathbf{Q}[x]/(f)$	<code>polgalois(f)</code>
initializes a Galois group structure $G$	<code>galoisinit(pol, {den})</code>
character table of $G$	<code>galoischartable(G)</code>
conjugacy classes of $G$	<code>galoisconjclasses(G)</code>
$\det(1 - \rho(g)T)$ , $\chi$ character of $\rho$	<code>galoischarpoly(G, \chi, {o})</code>
$\det(\rho(g))$ , $\chi$ character of $\rho$	<code>galoischarpoly(G, \chi, {o})</code>
action of $p$ in nfgaloisconj form	<code>galoispermtpol(G, {p})</code>
identify as abstract group	<code>galoisidentify(G)</code>
export a group for GAP/MAGMA	<code>galoisexport(G, {flag})</code>
subgroups of the Galois group $G$	<code>galoisissubgroups(G)</code>
is subgroup $H$ normal?	<code>galoisisnormal(G, H)</code>
subfields from subgroups	<code>galoissubfields(G, {flag}, {v})</code>
fixed field	<code>galoisfixedfield(G, perm, {flag}, {v})</code>
Frobenius at maximal ideal $P$	<code>idealfrobenius(nf, G, P)</code>
ramification groups at $P$	<code>idealramgroups(nf, G, P)</code>
is $G$ abelian?	<code>galoisisabelian(G, {flag})</code>
abelian number fields/ $\mathbf{Q}$	<code>galoissubcyclo(N, H, {flag}, {v})</code>

**Algebraic Number Theory**

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**The galpol package**

query the package: polynomial	<code>galoisgetpol(a, b, {s})</code>
... : permutation group	<code>galoisgetgroup(a, b)</code>
... : group description	<code>galoisgetname(a, b)</code>

**Relative Number Fields (rnf)**

Extension $L/K$ is defined by $T \in K[x]$ .	
absolute equation of $L$	<code>rnfequation(nf, T, {flag})</code>
is $L/K$ abelian?	<code>rnfisabelian(nf, T)</code>
relative nfaltobasis	<code>rnfaltobasis(rnf, x)</code>
relative nfbasistoalg	<code>rnfbasistoalg(rnf, x)</code>
relative <b>idealhnf</b>	<code>rnfidealhnf(rnf, x)</code>
relative <b>idealmul</b>	<code>rnfidealmul(rnf, x, y)</code>
relative <b>idealtwoelt</b>	<code>rnfidealtwoelt(rnf, x)</code>

**Lifts and Push-downs**

absolute $\rightarrow$ relative representation for $x$	<code>rnfeltabstorel(rnf, x)</code>
relative $\rightarrow$ absolute representation for $x$	<code>rnfeltreltoabs(rnf, x)</code>
lift $x$ to the relative field	<code>rnfeltup(rnf, x)</code>
push $x$ down to the base field	<code>rnfeltdown(rnf, x)</code>
idem for $x$ ideal: ( <b>rnfideal</b> )reltoabs, abstorel, up, down	

**Norms and Trace**

relative norm of element $x \in L$	<code>rnfeltnrm(rnf, x)</code>
relative trace of element $x \in L$	<code>rnfelttrace(rnf, x)</code>
absolute norm of ideal $x$	<code>rnfidealnrmabs(rnf, x)</code>
relative norm of ideal $x$	<code>rnfidealnrmrel(rnf, x)</code>
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$	<code>bnfisintnorm(bnf, x)</code>
is $x \in \mathbf{Q}$ a norm from $K$ ?	<code>bnfisnorm(bnf, x, {flag})</code>
initialize $T$ for norm eq. solver	<code>rnfisnorminit(K, pol, {flag})</code>
is $a \in K$ a norm from $L$ ?	<code>rnfisnorm(T, a, {flag})</code>
initialize $t$ for Thue equation solver	<code>thueinit(f)</code>
solve Thue equation $f(x, y) = a$	<code>thue(t, a, {sol})</code>
characteristic poly. of $a \bmod T$	<code>rnfcharpoly(nf, T, a, {v})</code>

**Factorization**

factor ideal $x$ in $L$	<code>rnfidealfactor(rnf, x)</code>
$[S, T]: T_{i,j} \mid S_i; S$ primes of $K$ above $p$	<code>rnfidealprimedec(rnf, p)</code>

**Maximal order  $\mathbf{Z}_L$  as a  $\mathbf{Z}_K$ -module**

relative polredbest	<code>rnfpolredbest(nf, T)</code>
relative polredabs	<code>rnfpolredabs(nf, T)</code>
relative Dedekind criterion, prime $pr$	<code>rnfdedekind(nf, T, pr)</code>
discriminant of relative extension	<code>rnfdisc(nf, T)</code>
pseudo-basis of $\mathbf{Z}_L$	<code>rnfpsseudobasis(nf, T)</code>

<b>General <math>\mathbf{Z}_K</math>-modules:</b> $M = [\text{matrix, vec. of ideals}] \subset L$	
relative HNF / SNF	<code>nfhnf(nf, M), nfsnf</code>
multiple of $\det M$	<code>nfdetint(nf, M)</code>
HNF of $M$ where $d = nfdetint(M)$	<code>nfhnfmod(x, d)</code>
reduced basis for $M$	<code>rnflllgram(nf, T, M)</code>
determinant of pseudo-matrix $M$	<code>rnfdet(nf, M)</code>
Steinitz class of $M$	<code>rnfsteinitz(nf, M)</code>
$\mathbf{Z}_K$ -basis of $M$ if $\mathbf{Z}_K$ -free, or 0	<code>rnfhnfbasis(bnf, M)</code>
$n$ -basis of $M$ , or $(n + 1)$ -generating set	<code>rnfbasis(bnf, M)</code>
is $M$ a free $\mathbf{Z}_K$ -module?	<code>rnfisfree(bnf, M)</code>

**Associative Algebras**

$A$  is a general associative algebra given by a multiplication table  $mt$  (over  $\mathbf{Q}$  or  $\mathbf{F}_p$ ); represented by  $al$  from **algtblinit**.  
create  $al$  from  $mt$  (over  $\mathbf{F}_p$ ) `algtblinit(mt, {p = 0})`  
group algebra  $\mathbf{Q}[G]$  (or  $\mathbf{F}_p[G]$ ) `alggroup(G, {p = 0})`  
center of group algebra `alggroupcenter(G, {p = 0})`

**Properties**

is $(mt, p)$ OK for <b>algtblinit</b> ?	<code>algisassociative(mt, {p = 0})</code>
multiplication table $mt$	<code>algmtable(al)</code>
dimension of $A$ over prime subfield	<code>algdim(al)</code>
characteristic of $A$	<code>algchar(al)</code>
is $A$ commutative?	<code>algiscommutative(al)</code>
is $A$ simple?	<code>algissimple(al)</code>
is $A$ semi-simple?	<code>algissemisimple(al)</code>
center of $A$	<code>algcenter(al)</code>
Jacobson radical of $A$	<code>algradical(al)</code>
radical $J$ and simple factors of $A/J$	<code>algsimpledec(al)</code>

**Operations on algebras**

create $A/I, I$ two-sided ideal	<code>algquotient(al, I)</code>
create $A_1 \otimes A_2$	<code>algtensor(al1, al2)</code>
create subalgebra from basis $B$	<code>algsubalg(al, B)</code>
quotients by ortho. central idempotents $e$	<code>algcentralproj(al, e)</code>
isomorphic alg. with integral mult. table	<code>algmakeintegral(mt)</code>
prime subalgebra of semi-simple $A$ over $\mathbf{F}_p$	<code>algprimesubalg(al)</code>
find isomorphism $A \cong M_d(\mathbf{F}_q)$	<code>algsplit(al)</code>

**Operations on lattices in algebras**

lattice generated by cols. of $M$	<code>alglathnf(al, M)</code>
... by the products $xy, x \in lat1, y \in lat2$	<code>alglatmul(al, lat1, lat2)</code>
sum $lat1 + lat2$ of the lattices	<code>alglatadd(al, lat1, lat2)</code>
intersection $lat1 \cap lat2$	<code>alglatinter(al, lat1, lat2)</code>
test $lat1 \subset lat2$	<code>alglatsubset(al, lat1, lat2)</code>
generalized index $(lat2 : lat1)$	<code>alglatindex(al, lat1, lat2)</code>
$\{x \in al \mid x \cdot lat1 \subset lat2\}$	<code>alglatlefttransporter(al, lat1, lat2)</code>
$\{x \in al \mid lat1 \cdot x \subset lat2\}$	<code>alglatrighttransporter(al, lat1, lat2)</code>
test $x \in lat$ (set $c = \text{coord. of } x$ )	<code>alglatcontains(al, lat, x, {\&amp;c})</code>
element of $lat$ with coordinates $c$	<code>alglatelement(al, lat, c)</code>

**Operations on elements**

$a + b, a - b, -a$	<code>algadd(al, a, b), algsub, algneg</code>
$a \times b, a^2$	<code>algmul(al, a, b), algsqrt</code>
$a^n, a^{-1}$	<code>algpow(al, a, n), alginv</code>
is $x$ invertible ? (then set $z = x^{-1}$ )	<code>algisinv(al, x, {\&amp;z})</code>
find $z$ such that $x \times z = y$	<code>algdivl(al, x, y)</code>
find $z$ such that $z \times x = y$	<code>algdivr(al, x, y)</code>
does $z$ s.t. $x \times z = y$ exist? (set it)	<code>algisdivl(al, x, y, {\&amp;z})</code>
matrix of $v \mapsto x \cdot v$	<code>algtomatrix(al, x)</code>
absolute norm	<code>algnorm(al, x)</code>
absolute trace	<code>algtrace(al, x)</code>
absolute char. polynomial	<code>algcharpoly(al, x)</code>
given $a \in A$ and polynomial $T$ , return $T(a)$	<code>algpoleval(al, T, a)</code>
random element in a box	<code>algrandom(al, b)</code>

Based on an earlier version by Joseph H. Silverman

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Send comments and corrections to (Karim.Belabas@math.u-bordeaux.fr)

Central Simple Algebras

$A$  is a central simple algebra over a number field  $K$ ; represented by  $al$  from **alginit**;  $K$  is given by a  $nf$  structure.  
create CSA from data           **alginit**( $B, C, \{v\}, \{maxord = 1\}$ )  
multiplication table over  $K$             $B = K, C = mt$   
cyclic algebra ( $L/K, \sigma, b$ )            $B = rnf, C = [sigma, b]$   
quaternion algebra  $(a, b)_K$             $B = K, C = [a, b]$   
matrix algebra  $M_d(K)$             $B = K, C = d$   
local Hasse invariants over  $K$     $B = K, C = [d, [PR, HF], HI]$

Properties

type of  $al$  ( $mt, CSA$ )           **algtype**( $al$ )  
dimension of  $A$  over  $\mathbf{Q}$            **algdim**( $al, 1$ )  
dimension of  $al$  over its center  $K$    **algdim**( $al$ )  
degree of  $A$  ( $= \sqrt{\dim_K A}$ )       **algdegree**( $al$ )  
 $al$  a cyclic algebra ( $L/K, \sigma, b$ ); return  $\sigma$    **algaut**( $al$ )  
...return  $b$                    **algb**( $al$ )  
...return  $L/K$ , as an  $rnf$        **algsplittingfield**( $al$ )  
split  $A$  over an extension of  $K$        **algsplittingdata**( $al$ )  
splitting field of  $A$  as an  $rnf$  over center   **algsplittingfield**( $al$ )  
multiplication table over center       **algrelmultable**( $al$ )  
places of  $K$  at which  $A$  ramifies       **algramifiedplaces**( $al$ )  
Hasse invariants at finite places of  $K$    **alghassef**( $al$ )  
Hasse invariants at infinite places of  $K$    **alghassei**( $al$ )  
Hasse invariant at place  $v$            **alghasse**( $al, v$ )  
index of  $A$  over  $K$  (at place  $v$ )       **algindex**( $al, \{v\}$ )  
is  $al$  a division algebra? (at place  $v$ )   **algisdivision**( $al, \{v\}$ )  
is  $A$  ramified? (at place  $v$ )       **algisramified**( $al, \{v\}$ )  
is  $A$  split? (at place  $v$ )           **algissplit**( $al, \{v\}$ )

Operations on elements

reduced norm                   **algnorm**( $al, x$ )  
reduced trace                  **algtrace**( $al, x$ )  
reduced char. polynomial       **algcharpoly**( $al, x$ )  
express  $x$  on integral basis    **algalgtobasis**( $al, x$ )  
convert  $x$  to algebraic form    **algbasistoalg**( $al, x$ )  
map  $x \in A$  to  $M_d(L)$ ,  $L$  split. field   **algtomatrix**( $al, x$ )

Orders

**Z**-basis of order  $\mathcal{O}_0$            **algbasis**( $al$ )  
discriminant of order  $\mathcal{O}_0$        **algdisc**( $al$ )  
**Z**-basis of natural order in terms  $\mathcal{O}_0$ 's basis   **alginvbasis**( $al$ )